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Quantum Theory of a Friedmann Field

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A quantum theory of a generalized closed Friedmann model, with a dust of negative "masses" filling the universe, is constructed. It is shown that the second quantization procedure allows one to avoid having to interpret the dust mass as negative on the classical level of description. A difficulty in introducing a probability character to the wave function of the universe also is solved. Finally, after realizing this standard method of describing physical fields used in the theory of elementary particles, a quantum theory of Friedmann universes is constructed.

Einstein theory of gravity and quantum mechanics have strongly influenced our world outlook. The unification of these theories, quantum geometrodynamics, is still under construction. Each stage is obtained by hard work demanding, first of all, development of one's limited human imagination. In a quarter century of active elaboration of quantum geometrodynamics beginning with the paper of De Witt (1967), some of the mysterious aspects of this undertaking have been probed. In particular, many papers, including the recent important publications of Vilenkin (1989), Hosoya and Morikawa (1989), and Kiefer and Singh (1991), have treated the interpretation of the wave function obeying the Wheeler–De Witt functional differential equation

$$\left[-\frac{16\pi G\hbar^2}{c^2}G_{ijkl}(\mathbf{r})\frac{\delta^2}{\delta\gamma_{ij}(\mathbf{r})\delta\gamma_{kl}(\mathbf{r})}-\frac{c^4}{16\pi G}\sqrt{\gamma}\,{}^{(3)}R + \hat{H}_m(\gamma_{ij}(\mathbf{r}),\Phi(\mathbf{r}))\right]\Psi[\gamma_{ij}(\mathbf{r}),\Phi(\mathbf{r})]=0 \tag{1}$$

Here $\gamma_{ij}(\mathbf{r})$ is the metric of 3-space, γ is the determinant of the metric tensor, through $\Phi(\mathbf{r})$ symbolically marked in materials fields ($\Phi(\mathbf{r})$ is some field

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interacting with gravity), ${}^{(3)}R$ is the Ricci scalar of 3-space, and G_{ijkl} is the metric of the superspace,

$$G_{ijkl} = \frac{1}{2}\gamma^{-1/2}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}) \tag{2}$$

This paper will try to clear up what Ψ means on the basis of isotropic universes.

The 6-dimensional configuration space $\{\gamma_{ij}\}$ has hyperbolic signature. The Laplace–Beltrami operator (1) is defined on this space.

Now it does not seem surprising that the wave function does not depend on a time coordinate t; by virtue of this one would think there is no dynamics of the gravitational field. In general relativity time is introduced as an arbitrary coordinate (mark) on which the physics of phenomena does not depend, but the true physically meaningful time is defined from intrinsic geometrical and material characteristics of the investigated system (see, e.g., Pavlov, 1993). The universe's time and space characteristics were born together with the universe itself and it is necessary to search for them among the coordinates of the superspace—the arena of the development of the world.

The following legitimate question arises: The Wheeler–De Witt equation for gravity without matter being of hyperbolic type (which will also be true in our investigated models), is it possible to construct from the conservation law for the probability current a nonnegative-definite quantum mechanical probability density? A similar problem faced physicists at the dawn of the development of quantum mechanics.

We consider models of isotropic universes filled with an elastic gas with the compressibility law

$$\epsilon = c^2\rho + \alpha\rho^n \tag{3}$$

where ϵ is a volume density of the energy of matter, c is the speed of light, ρ is the volume density of matter, α is a constant, and n is a real parameter, $4/3 \leq n \leq 2$. For simplicity in writing the formulas, we put $16\pi G/3 = 1$, $c = 1$, and $\hbar = 1$ and normalize the functional of the action for the volume V:

$$V = \int \begin{bmatrix} \sin^2\chi \\ \mathrm{sh}^2\chi \\ \chi^2 \end{bmatrix} \sin\vartheta \, d\chi \, d\vartheta \, d\varphi$$

combining all these cases of isotropic models. Then the functional takes the form

$$S = \frac{1}{2}\int d\eta \left(-\frac{\dot{a}^2}{N} + Nka^2\right) + \frac{n-1}{n(n\alpha)^{1/n-1}}\int d\eta\, Na^{(3n-4)/(n-1)}\left(\frac{\dot{\sigma}}{N} - a\right)^{n/(n-1)} \tag{4}$$

where η is a parameter, N is a lapse function, a is a scale of the universe, $k = 1$ for a closed model, $k = 0$ for a flat one, and $k = -1$ for an open one, and $\dot{\sigma}/(Na)$ is the mass density of the enthalpy of the gas. It is interesting that the case $n = 4/3$, as is seen from (4), stands out and corresponds to the most physically important situation when in the case of strongly compressed matter the equation of state of the gas is of ultrarelativistic type and in the case of weakly compressed matter it is of dust-like type. Later we put $n = 4/3$.

The Hamiltonian constraint takes the form

$$\varphi_0 = -\frac{1}{2}p_a^2 - \frac{k}{2}a^2 + \alpha p_\sigma^{4/3} + ap_\sigma \tag{5}$$

where p_a and p_σ are variables canonically conjugate to a and σ, and p_σ is the mass of the dust of the universe (Burlankov *et al.*, 1984). Let us substitute the parameter A for α: $\alpha \equiv (3/4)A^{2/3}$.

In Burlankov *et al.* (1984) and Pavlov (1992a,b) physical considerations led to consideration of nonnegative values of p_σ only. This led to the consistency of the constructed quantum theories with the classical solutions: the quantum average values of the wave packets move along the classical trajectories consistent with the Ehrenfests theorem. But as there are no requirements in principle for excluding negative masses in general relativity (Bondi, 1957), we expand the interval of p_σ to $-\infty < p_\sigma < +\infty$ in this paper. The symbol p_σ denotes a value like the mass of a substance. It can be called a mass; however, we have to bear in mind that the concept of the mass of a completely closed universe absolutely has no reasonable definite physical meaning (Misner *et al.*, 1973). There is no platform outside of the universe where we could measure its mass by its interactions, or study periods of some Kepler orbits, or by some other method. Refusing the attractive quantum mechanical conceptions that allowed the principle of correspondence, it is possible to construct a correct quantum theory of a Friedmann field with a positive spectrum of particle masses.

For the closed and the open models one takes a canonical univalent mapping $(a, p_a; \sigma, p_\sigma) \mapsto (x, p_x; t, p_t)$:

$$x = a - kp_\sigma, \qquad t = \sigma - kp_a$$
$$p_x = p_a, \qquad p_t = p_\sigma \tag{6}$$

The Hamilton equations after the transformation are

$$\frac{dx}{d\eta} = -p_x, \qquad \frac{dt}{d\eta} = p_t + (A^2 p_t)^{1/3}$$

$$\frac{dp_x}{d\eta} = kx, \qquad \frac{dp_t}{d\eta} = 0$$

$$\frac{1}{2} p_x^2 + \frac{k}{2} x^2 = \frac{1}{2} p_t^2 + \frac{3}{4} (A p_t^2)^{2/3} \tag{7}$$

The structure of the equations is quite clear: we have to deal with the expanded phase system of an oscillator (antioscillator) where t is the true time and p_t is the Hamiltonian canonically conjugate to the time t. But now there is also included into the classical mechanics a movement of particles with a negative energy p_t and an oppositely directed arrow of time t.

Let us go to the quantum level of description of the system in Dirac's approach and consider the case $k = 1$. The quantum Wheeler–De Witt equation after the Fourier transformation

$$\psi(t, x) = \int dp_t \exp(itp_t)\psi(p_t, x) \tag{8}$$

which in the classical language of description also corresponds to a univalent mapping, is

$$\left[-\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - E(p_t) \right] \psi(p_t, x) = 0 \tag{9}$$

where the notation for the function $E(p_t)$ is introduced:

$$E(p_t) \equiv \frac{1}{2} p_t^2 + \frac{3}{4} (A p_t^2)^{2/3} \tag{10}$$

corresponding to the massless particle of Klein–Gordon–Fock in the field of the harmonic oscillator. Representing the solution of (9) in terms of Hermite polynomials, we get

$$\psi(t, x) \sim \int dp_t \exp(itp_t) \sum_{n=0}^{\infty} a_n(p_t) \exp(-x^2/2)\, H_n(x)\, \delta(E(p_t) - \epsilon_n) \tag{11}$$

where $a_n(p_t)$ are complex coefficients and $\epsilon_n \equiv n + 1/2$ is the spectrum of the harmonic oscillator. One calculates the integral using the representation of the δ-function

$$\delta(E(p_t) - \epsilon_n) = [\delta(p_t - p_t(\epsilon_n)) + \delta(p_t + p_t(\epsilon_n))] / |\partial E(p_t)/\partial p_t|_{\epsilon_n}$$

and gets the solution of the quantum problem:

$$\psi(t, x) = \sum_{n=0}^{\infty} \left(\frac{1}{2^n n! \sqrt{\pi}}\right)^{1/2} H_n(x) \frac{\exp(-x^2/2)}{(2|E_n|)^{1/2}}$$

$$\times [b_n^*(E_n) \exp(iE_n t) + a_n(E_n) \exp(-iE_n t)] \qquad (12)$$

where $E_n \equiv p_t(\epsilon_n)$ are positive roots of the algebraic equation

$$\frac{1}{2} E_n^2 + \frac{3}{4} (AE_n^2)^{2/3} = n + 1/2 \qquad (13)$$

So the energy spectrum of the system is not equidistant.

The Born interpretation of the ψ-function turns out to be infeasible, as follows already from the two-valuedness of the function $E^{-1} = E^{-1}(p_t)$. The solution combines positive-frequency and negative-frequency modes. We consider the ψ-function as complex ($a_n \neq b_n$), without restricting the content of the model.

Let us consider for simplicity the case $A \to 0$, i.e., the Friedmann model. Then the energy spectrum of the states is $E_n = (2n + 1)^{1/2}$. To investigate the dynamical characteristics of the solutions, we build the functional of the action

$$S = \int dt\, dx\, \mathcal{L} = \int dt\, dx \left(\frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial x} \frac{\partial \psi}{\partial x} - \psi^* x^2 \psi\right) \qquad (14)$$

which we minimize on the class of the functions (12):

$$\psi(t, x) = \sum_{n=0}^{\infty} u_n(x) \frac{1}{(2|E_n|)^{1/2}} [b_n^*(E_n) \exp(iE_n t) + a_n(E_n) \exp(-iE_n t)]$$

We find the Hamiltonian H, introducing the momentum density π:

$$H = \int dx\, (\pi^* \dot{\psi}^* + \pi \dot{\psi} - \mathcal{L}) = \sum_{n=0}^{\infty} E_n (b_n b_n^* + a_n^* a_n) \qquad (15)$$

The equation of continuity will be obtained from the equations of motion

$$\frac{\delta S}{\delta \psi} = 0, \qquad \frac{\delta S}{\delta \psi^*} = 0$$

$$\frac{\partial}{\partial t} [i(\psi^* \partial_t \psi - \psi \partial_t \psi^*)] + \frac{\partial}{\partial x} \left[\frac{1}{i} (\psi^* \partial_x \psi - \psi \partial_x \psi^*)\right] = 0 \qquad (16)$$

Let us calculate the charge $Q = \int dx\, \rho(x)$:

$$Q = \sum_{n=0}^{\infty} (a_n^* a_n - b_n^* b_n) \qquad (17)$$

The one-time commutation relations

$$[\psi(t, x), \pi(t, y)] = i\delta(x - y)$$

are satisfied if

$$[a_n, a_m^+] = [b_n, b_m^+] = \delta_{nm} \tag{18}$$

and all other pairs of operators commute. Using the algebraic relations (18), it is not difficult to deduce that the operators a_n^+ and a_n create and destroy the quanta of the field a, and the operators b_n^+ and b_n create and destroy those of the field b. The operators

$$N_n \equiv a_n^+ a_n, \qquad \overline{N}_n \equiv b_n^+ b_n$$

commute with the Hamiltonian and between themselves. They are the operators of the number of particles and antiparticles, so with their help one can construct the basis of the quantum states. The vacuum state $|0\rangle$ does not contain particles and antiparticles at all. It corresponds to the state "nothing" when space and time are absent. The Fock space of the universes and the antiuniverses are made of the vectors of the states

$$\begin{aligned} |n_1, n_2, \ldots, \overline{n}_1, \overline{n}_2, \ldots\rangle = \frac{1}{(n_1! n_2! \ldots)^{1/2}} [a_1^+]^{n_1} [a_2^+]^{n_2} \cdots \\ \times [b_1^+]^{\overline{n}_1} [b_2^+]^{\overline{n}_2} \cdots |0\rangle \end{aligned} \tag{19}$$

which corresponds to bosons. The operator of the Hamiltonian (15) after second quantization takes the form

$$H = \sum_{n=0}^{\infty} E_n (a_n^+ a_n + b_n^+ b_n + 1) = \sum_{n=0}^{\infty} E_n (N_n + \overline{N}_n + 1) \tag{20}$$

Its eigenvalues are positive. So the problem of "negative energies" (negative masses of the dust p_0) is solved. The energy of the vacuum, as in the theory of elementary particles, is infinite. The charge Q of the field of the universes and antiuniverses

$$Q = \sum_{n=0}^{\infty} (N_n - \overline{N}_n) \tag{21}$$

commutes with the Hamiltonian H and so is conserved and by definition is equal to the difference between the number of worlds and the number of antiworlds. A world and an antiworld differ only by a certain quantum number which could have an arbitrary nature; we called it a charge. A world carries a charge $+1$, the charge of an antiworld is -1. Both enter into the theory symmetrically: they coexist. In order to distinguish them physically, it is necessary to introduce an interaction between them.

Returning to the wave function of the universe (12), which is the operator of creating antiparticles and destroying particles, we underline that the question of the localization of universes in the Wheeler–De Witt superspace now has no physical meaning. The departure from the probability interpretation rules out such inappropriate questions.

We notice yet that from an abstract contemplator's point of view measuring his time with the parameter η, every universe has its own clock. The velocity of the flow of a universe's time depends on the mass of the dust (7), i.e., the many-times formalism holds (Wentzel, 1949).

The dramatic peculiarity of the introduced representation is a multitude of universes and antiuniverses: They can be created and annihilated, if the difference between their numbers is constant, in accordance with the laws of the quantum field theory formalism. We are used to a change of numbers of elementary particles in high-energy physics: acts creating and annihilating particles are ordinary events registered in physical laboratories. But the transference of the formalism of second quantization into quantum cosmology demands inevitably the modification of the Copenhagen doctrine about the quantum nature of matter. In quantum geometrodynamics $\Psi(q)$ describes not only the physical state of the system, but also the state of the hypersurface, i.e., the observer himself. What is more, after using second quantization we come to a many-worlds conception. The philosophical basis of such unusual representations (Everett's interpretation of quantum gravitation) was prepared in the 1950s (Everett, 1957; Wheeler, 1957). But such problems are quite new to physics because the object of investigation can be presented by processes taking place in another universe also existing in reality but topologically not connected with ours.

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